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An Unexpected Result in Coding the Vertices of a Graph

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I. INTRODUCTION

Let G be a graph having n vertices. Two distinct nodes are said to be adjacent if and only if they are joined by a single branch, and every vertex is considered to be adjacent to itself. If there exists two vertices in G which cannot be joined by going from one adjacent node to another, then G is said to be nonconnected; otherwise it is connected. If all vertices are adjacent, then G is said to be completely connected.

Associate with each vertex i of G a unique binary code c_i (sometimes written $c(i)$) of length m. If $c_i = (x_{i1}x_{i2}, \ldots x_{im})$ where $x_{ik} \in \{0, 1\}$, then the Hamming distance between c_i and c_j is

$$H(c_i, c_j) = \sum_{k=1}^{m} | x_{ik} - x_{jk} | .$$

Finally, let T be a positive integer threshold value, $\| c_i \| = \sum_{k=1}^{m} x_{ik}$, and c_ic_j the concatenation of the two codes c_i and c_j.

The coding problem can be stated as follows. G is codable (more specifically $[T, m]$ codable) if there exists an m and T such that $H(c_i, c_j) \leqslant T$ if and only if vertices i and j are adjacent. T is selected as the smallest integer which satisfies this inequality. G is T codable means that there exists an m such that G is $[T, m]$ codable, and G is m codable means that there exists a T such that G is $[T, m]$ codable.

The following results have previously been shown by Breuer [1]:

(a) Every graph is codable

(b) For every value of T there exists a graph which is not T codable

(c) If G is T' codable, then it is also T codable for $T = pT'$ and $T = T' + 2p$, for $p = 1, 2, 3,\ldots$.

(d) If G is T' codable, then G is codable for all $T \geqslant 2T'$ if T' is odd, otherwise for all $T' \geqslant 2T' + 1$ if T' is even.

(e) The completely connected graph is $T = 2$ codable.

In this paper we will extend some of these results. A few self-evident results are:

(f) For every value of T and m, where $1 \leqslant T \leqslant m$, $m > 1$, and $n \leqslant 2^m$, there exists a graph which is not $[T, m]$ codable.

To prove this result, note that for $T < m$, the completely connected graph having 2^m vertices is not $[T, m]$ codable. For $T = m$, any noncompletely connected graph is not $[T, m]$ codable.

Hence $[T, m]$ codability does not imply $[T + 1, m]$ codability, and we have

(g) For each value of T and m ($T \leqslant m$), there exists a graph which is $[T, m]$ codable but not $[T \pm 1, m]$ codable.

To prove this, let Q_m be the set of all (0, 1) m-tuples and let $n = 2^m$. Let c be a one-to-one assignment of vertices to elements in Q_m. Now join vertices u and v if and only if $H(c(u), c(v)) \leqslant T$. The graph so formed is $[T, m]$ codable but not $[T \pm 1, m]$ codable.

(h) If G is T codable, then any subgraph of G is T codable.

If G' is a subgraph of G, and c is a T coding of G, then the restriction of c to the vertices of G' produces a T coding for G'.

II. The Codability of Graphs

We now give a new proof to result (a). The proof is constructive, and produces an upper bound on the minimal values of T and m. Assume that in G every vertex is adjacent to at most P other vertices. We then have

Theorem 1. *Given a graph G with n vertices, and where each vertex is adjacent to at most P ($n - 1 \geqslant P \geqslant 2$) other vertices. Then G is $[T, m]$ codable where $T = 4P - 4$ and $m = 2Pn$. Hence G is T' codable for some $T' \leqslant 4P - 4$ and is m' codable for some $m' \leqslant 2Pn$. (If $P = 0$ or 1, it is easy to show that a $[1, 2n]$ coding exists.)*

Proof. We construct a code such that G is $[T, m]$ codable for $T = 4P - 4$ and $m = 2Pn$. The code will be partitioned into n blocks of $2P$ bits each, with block j corresponding to vertex j. Block j of code i is further partitioned into two groups $a_j{}^i$ and $b_j{}^i$, each of P bits. We have

$$c_i = (a_1{}^i b_1{}^i a_2{}^i b_2{}^i \cdots a_n{}^i b_n{}^i),$$

where

$$a_j{}^i = a^i_{j1} a^i_{j2} \cdots a^{i'}_{jP} \qquad \text{and} \qquad b_j{}^i = b^{i'}_{j1} b^{i'}_{j2} \cdots b^{i'}_{jP}\,.$$

Label each branch associated with vertex i. Label one branch i_1, another i_2,..., and label the P_ith branch i_{P_i}, where P_i is the number of branches connected to vertex i, and where $P_i \leqslant P$. This is done for all n vertices. Since each branch is connected to two vertices, each branch will be assigned two labels as illustrated below.

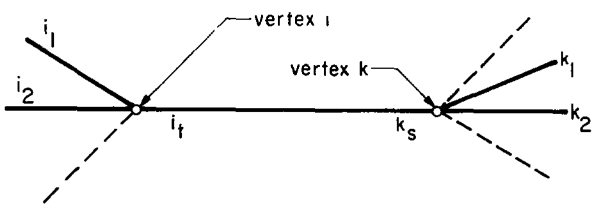


FIG. 1

Rule a: Set $b^i_{jk} = 1$ for $k = 1, 2,..., P$ and all $i = j$.

Rule b: If vertices i and k are nonadjacent, set $b^k_{ij} = b^i_{kj} = 0$ for $j = 1, 2,..., P$.

Rule c: If vertices i and k are adjacent, and the branches are labeled as shown above, set $b^i_{ks} = b^k_{it} = 1$, and the remainder of the b's in block k and i of vertices i and k are zero.

Apply rules b and c to all blocks j of vertex i for $j \neq i$, and for all i. The groups $a_j{}^i$ are filled in a follows.

Rule d: If vertex i is adjacent to P_i vertices (not including itself), place $P - P_i$ *ones* and P_i *zeros* into block $a_i{}^i$. Usually the $P - P_i$ *ones* are placed into the first $P - P_i$ positions of $a_i{}^i$. Finally, set $a^i_{jk} = 0$ for $k = 1, 2,..., P$, $i = 1, 2,..., n$, and for all $j \neq i$.

We now show that this code has the properties claimed. First, each code contains $2P$ *one* elements and $2P(n - 1)$ *zero* elements. If vertices i and k are adjacent, then due to rule c there exists at least two positions such that both codes c_i and c_k have *ones* in these two positions, hence $H(c_i, c_k) \leqslant 4P - 4$.

If vertices i and k are nonadjacent, then rule b applies. Vertices i and k can be adjacent to some common vertex p, but they will be connected to p by a unique branch labeled p_s and p_t, where $s \neq t$. Hence $b^i_{ps} = b^k_{pt} = 1$ and $H(b_p{}^i, b_p{}^k) = 2$, i.e., the *one* bits are in different positions and do not reduce the Hamming distance. Finally, the *one* bits introduced by rule *a* are each in a unique bit position, hence we have

$$H(c_i, c_k) = 4P - 4 = T \text{ if vertices } i \text{ and } k \text{ are adjacent}$$
$$H(c_i, c_k) = 4P = T + 4 \text{ if vertices } i \text{ and } k \text{ are nonadjacent}$$
$$m = 2Pn$$

and

$$\| c_i \| = 2P$$

Example:

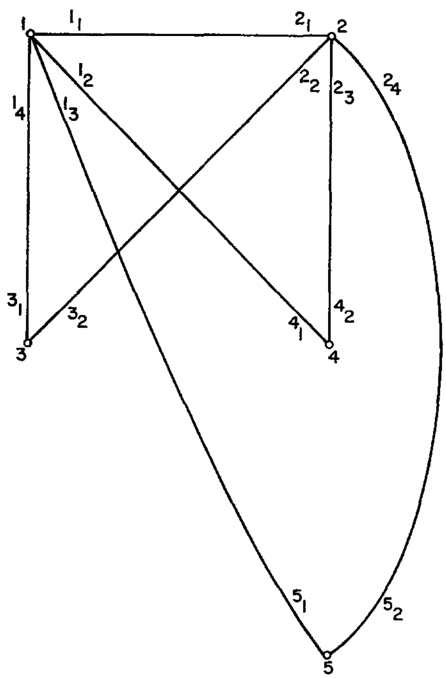


FIG. 2

Given the graph G shown above with $n = 5$ and $P = 4$. The branches are all labeled. The code c_3 is determined as follows. Vertices 1 and 3 are adjacent, so according to rule *c* we have $b_1^3 = (0001)$. The *one* is in the fourth position because the branch connecting the two vertices is labeled 1_4. Vertices 2 and 3 are connected by branch 2_2, hence $b_2^3 = (0100)$. Vertices 4 and 5 are not adjacent to vertex 3, hence, according to rule b we have $b_4^3 = b_5^3 = (0000)$. From rule a we have $b_3^3 = (1111)$. Since vertex 3 is adjacent to two vertices, we place $4 - 2 = 2$ *ones* in group a_3^3 (rule d). Arbitrarily we place these *ones* into the first two bit positions, i.e., $a_3^3 = (1100)$ The remaining a_j^3 ($j \neq 3$) groups are set to *zero*.

The entire code is shown below.

```
       ****      ****       **        **        **
c1 = (0000 1111 0000 1000 0000 1000 0000 1000 0000 1000)
c2 = (0000 1000 0000 1111 0000 0100 0000 0100 0000 0100)
c3 = (0000 0001 0000 0100 1100 1111 0000 0000 0000 0000)
c4 = (0000 0100 0000 0010 0000 0000 1100 1111 0000 0000)
c5 = (0000 0010 0000 0001 0000 0000 0000 0000 1100 1111)
       a1   b1   a2   b2   a3   b3   a4   b4   a5   b5
```

Note that $H(c_i, c_j) = 12$ if vertices i and j are adjacent ($i \neq j$), otherwise $H(c_i, c_j) = 16$.

We now show that for the case $P \geqslant 2$ and $n \geqslant 2$, the bound $m' \leqslant 2P_n$ given by Theorem 1 is not a least upper bound.

To show this we note that since $P \geqslant 2$ there exist a branch (actually at least two) in the graph, say branch i_α. Hence group a_i^i has at least one *zero* entry, say a_{iP}^i Now $a_{iP}^j = 0$ for all j including $i = j$, hence bit position a_{iP}^i can be deleted without affecting the required properties of the code. In fact, all bit positions which are *zero* in all code words can be deleted. If there are a total of N branches in the graph, then there are $2N$ such positions which can be deleted. These are indicated in the example ($N = 7$) by an asterisk.

COROLLARY 1. *All graphs G of n nodes are $[T, m]$ codable for $T = 4(n - 2)$ and $m = 2(n - 1)\, n$.*

PROOF. This result follows from theorem 1 by setting

$$P = P(\text{max.}) = n - 1.$$

III. AN UNEXPECTED RESULT

In this section the results previously given in (d) are extended. We show that if G is T' codable, and T' is even, then G is T codable for all $T \geqslant T'$. Also, if G is T codable, and T is odd, then G is not necessarily $T + 1$ codable. Since these results are independent of m, we reformulate the definitions of codability as follows:

Let Q denote the set of all infinite sequences $x = (x_1, x_2, x_3, \ldots)$ where each $x_i = 0$ or 1 and all but finitely many of the x_i are zero. For $x, y \in Q$ let

$$H(x, y) = \sum_{i=1}^{\infty} | x_i - y_i | .$$

Let G be a (finite) graph. If T is a positive integer, then by a *T-coding* of G we mean a one-to-one function c from the vertices of G to Q which has the property that for any pair of vertices u and v in G, $H(c(u), c(v)) \leqslant T$ if and only if u and v are adjacent. We say that G is *T-codable* if G has a T-coding.

LEMMA 1. *If G is T-codable then G is $T + 2$ — codable and $2T$-codable. If T is even and G is T-codable then G is $T + 1$-codable.*

PROOF. The first statement is the same as result (c), and is proven in [1]. To prove the second statement, let the code associated with vertex u which makes the graph T-codable be $c(u)$. Let $c'(u) = \delta(u)\, c(u)$, where $\delta(u) \in \{0, 1\}$, and where $\delta(u) = 1$ if and only if $\| c(u) \|$ is odd, hence $\| c'(u) \|$ is even. Now $H(c'(u), c'(v)) \leqslant H(c(u), c(v)) + 1$, and if $H(c(u), c(v)) \geqslant T + 1$, then $H(c'(u), c'(v)) \geqslant T + 2$. Now $H(c'(u), c'(v))$ is even for all u, v pairs, and for some u, v pair, say x, y, we have that $H(c'(x), c'(y)) = T = 2k$. Let $c''(x) = 1c'(x)$, and $c''(u) = 0c'(u)$, for all $u \neq x$. Now $H(c''(x), c''(y)) = T + 1$, and the lemma is proved.

Since T-codability implies $T + 2$ codability and T codability implies $T + 1$ codability if T is even, we have that T codability implies T' codability for all $T' \geqslant T$ if T is even.

For each graph G let

$$T_0(G) = \min \{T \mid T \text{ is odd and } G \text{ is } T\text{-codable}\}$$

and

$$T_e(G) = \min \{T \mid T \text{ is even and } G \text{ is } T\text{-codable}\}.$$

From Lemma 1 it follows that a graph G is T-codable if and only if T is odd and $T \geqslant T_0(G)$ or T is even and $T \geqslant T_e(G)$.

THEOREM 2. *Let G be a graph. Then*

$$T_0(G) - 1 \leqslant T_e(G) \leqslant 2T_0(G). \tag{*}$$

Furthermore, if T_0 and T_e are positive integers which are respectively odd and even and if $T_0 - 1 \leqslant T_e \leqslant 2T_0$, then there is a graph G with $T_0(G) = T_0$ and $T_e(G) = T_e$.

PROOF. By Lemma 1, $T_e(G) \leqslant 2T_0(G)$ and $T_0(G) \leqslant T_e(G) + 1$. Combining these inequalities we get (*). To prove the second statement in the theorem we must first construct some graphs.

For each positive integer N, let $\mathscr{S}(N)$ be the set of all finite sequences $\alpha = (\alpha_1, \ldots, \alpha_\gamma)$ where each α_i is an integer with $1 \leqslant \alpha_i \leqslant N$. For $\alpha \in \mathscr{S}(N)$ let $|\alpha|$ denote the length of the sequence α. If $\alpha, \beta \in \mathscr{S}(N)$ we say $\alpha \leqslant \beta$ if $|\alpha| \leqslant |\beta|$ and $\alpha_i = \beta_i$ for $1 \leqslant i \leqslant |\alpha|$. If γ is any positive integer, we let $\mathscr{S}(\gamma, N)$ be the set of all sequences $\alpha \in \mathscr{S}(N)$ with $|\alpha| \leqslant \gamma$.

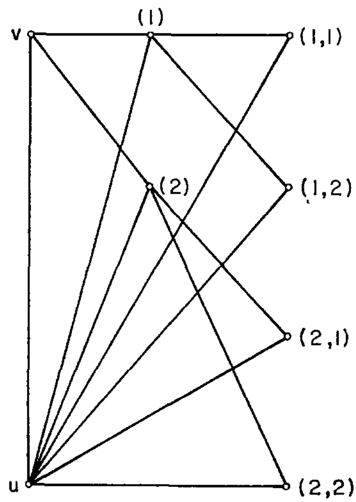
For each pair of positive integers γ and N let $G(\gamma, N)$ be the graph defined as follows: $G(\gamma, N)$ has as vertices the symbols u and v together with the elements of $\mathscr{S}(\gamma, N)$. The following adjacency relations hold:

u is adjacent to every vertex.

v is adjacent to every $\alpha \in \mathscr{S}(\gamma, N)$ with $|\alpha| = 1$.

$\alpha, \beta \in \mathscr{S}(\gamma, N)$ are adjacent if $\alpha \leqslant \beta$ and $|\alpha| + 1 = |\beta|$.

Fig. 2 illustrates the graph $G(2, 2)$.



S(2;2) = {(1), (2), (1,1), (1,2), (2,1), (2,2)}

FIG. 3: G(2,2).

LEMMA 2. *Let γ and N be positive integers with $\gamma > 1$. Let $\alpha \in \mathscr{S}(\gamma, N)$ with $|\alpha| = 1$. Let H be the subgraph of $G(\gamma, N)$ which is spanned by the vertex u and all vertices $\beta \in \mathscr{S}(\gamma, N)$ with $\beta \geqslant \alpha$. Then H is isomorphic to the graph $G(\gamma - 1, N)$. Furthermore, the isomorphism may be chosen so that the vertices u and α of H correspond to the vertices u and v of $G(\gamma - 1, N)$.*

PROOF. We define a function φ from the vertices of H to the vertices of $G(\gamma - 1, N)$ as follows: $\varphi(u) = u$ and $\varphi(\alpha) = v$. If $\beta \in S(\gamma, N)$ and $\beta \geqslant \alpha$ but $\beta \neq \alpha$, then $\beta = (\alpha_1, \beta_2, ..., \beta_k)$ where $k \geqslant 2$. We set $\varphi(\beta) = (\beta_2, ..., \beta_k)$. It is easily verified that φ is the desired isomorphism.

LEMMA 3. *Let γ, N and T be positive integers with $N \geqslant 2^T T + 1$. Suppose that c is a T-coding of $G(\gamma, N)$. Then*

$$H(c(u), c(v)) \leqslant T - \gamma \qquad \textit{if} \qquad T \textit{ is odd}$$

and

$$H(c(u), c(v)) \leqslant T - 2\gamma \qquad \textit{if} \qquad T \textit{ is even.}$$

PROOF. For each vertex x of $G(\gamma, N)$ let

$$I(x) = \{i \mid c_i(x) \neq c_i(u)\},$$

where $c_i(x)$ denotes the ith term of the sequence $c(x)$. If we let $| S |$ denote the cardinality of a set S, then $| I(x) | = H(c(x), c(u))$. Hence, for each $\alpha \in \mathscr{S}(1, N) \subset \mathscr{S}(\gamma, N)$ we have $1 \leqslant | I(\alpha) | \leqslant T$. Now

$$| \mathscr{S}(1, N) | = N \geqslant 2^T T + 1,$$

so, for some k with $1 \leqslant k \leqslant T$, there are at least $2^T + 1$ elements $\alpha \in \mathscr{S}(1, N)$ with $| I(\alpha) | = k$. For each such α, $I(\alpha) \cap I(v)$ is a subset of $I(v)$. Now $| I(v) | = H(c(v), c(u)) \leqslant T$ so $I(v)$ has at most 2^T distinct subsets. Hence, there are two elements $\alpha, \beta \in \mathscr{S}(1, N)$ with $| I(\alpha) | = | I(\beta) | = k$ and $I(\alpha) \cap I(v) = I(\beta) \cap I(v)$.

We will now argue by induction on γ. Suppose either that $\gamma = 1$ or that $\gamma > 1$ and that the lemma has been established for $\gamma - 1$. Let $\epsilon = 1$ if T is odd and let $\epsilon = 2$ if T is even. If $\gamma = 1$ then

$$k = | I(\alpha) | = H(c(\alpha), c(u)) \leqslant T = T - \epsilon(\gamma - 1).$$

On the other hand, if $\gamma > 1$ and Lemma 3 is true for $\gamma - 1$, then by combining Lemma 2, Lemma 3 for $\gamma - 1$, and the fact that a T-coding of $G(\gamma, N)$ restricted to a subset S of the vertices of $G(\gamma, N)$ is a T-coding of the subgraph spanned by S, we deduce that

$$T - \epsilon(\gamma - 1) \geqslant H(c(\alpha), c(u)) = | I(\alpha) | = k.$$

Hence, in either case we have

$$T - \epsilon(\gamma - 1) \geqslant k. \tag{1}$$

If x and y are vertices of $G(\gamma, N)$ then $c_i(x) \neq c_i(y)$ if and only if

$$i \epsilon (I(x) \cup I(y)) - (I(x) \cap I(y)).$$

Hence

$$\begin{aligned} H(c(x), c(y)) &= | (I(x) \cup I(y)) - (I(x) \cap I(y)) | \\ &= | I(x) | + | I(y) | - 2 | I(x) \cap I(y) | . \end{aligned}$$

Therefore,

$$T \geqslant H(c(\alpha), c(v)) = | I(\alpha) | + | I(v) | - 2 | I(\alpha) \cap I(v) | .$$

Letting $\ell = | I(\alpha) \cap I(v) |$ we have

$$T \geqslant k + | I(v) | - 2\ell. \tag{2}$$

Since $I(\alpha) \cap I(v) = I(\beta) \cap I(v)$ we have $I(\alpha) \cap I(v) \subset I(\alpha) \cap I(\beta)$. Therefore, $|I(\alpha) \cap I(\beta)| \geqslant \ell$. Consequently,

$$T < H(c(\alpha), c(\beta)) = |I(\alpha)| + |I(\beta)| - 2|I(\alpha) \cap I(\beta)| \leqslant 2k - 2\ell.$$

Since T is an integer and $2k - 2\ell$ is an even integer this strict inequality implies that

$$2k - 2\ell \geqslant T + \epsilon. \tag{3}$$

Adding (1) and (2) and applying (3) we get

$$2T - \epsilon(\gamma - 1) \geqslant 2k + |I(v)| - 2\ell \geqslant |I(v)| + T + \epsilon.$$

Therefore,

$$H(c(u), c(v)) = |I(v)| \leqslant T - \epsilon\gamma.$$

The lemma is now established by induction on γ.

If G is a graph, let $V(G)$ denote the set of vertices of G. Let G_1 and G_2 be graphs. By $G_1 \oplus G_2$ we mean the graph with

$$V(G_1 \oplus G_2) = (V(G_1) \times \{1\}) \cup (V(G_2) \times \{2\})$$

and with vertices $(x, i), (y, j) \in V(G_1 \oplus G_2)$ adjacent if and only if $i = j$ and x and y are adjacent in G_i. In other words, $G_1 \oplus G_2$ is just the disjoint union of G_1 and G_2.

If $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ we define $G_1 \oplus G_2/(v_1, v_2)$ as follows:

$$V(G_1 \oplus G_2/(v_1, v_2)) = ((V(G_1) - \{v_1\}) \times \{1\}) \cup ((V(G_2) - \{v_2\}) \times \{2\}) \cup \{0\}.$$

The following adjacency relations hold:

0 is adjacent to (x, i) if and only if v_i is adjacent to x in G_i.

(x, i) and (y, j) are adjacent if and only if $i = j$ and x and y are adjacent in G_i.

In other words, $G_1 \oplus G_2/(v_1, v_2)$ is obtained from $G_1 \oplus G_2$ by identifying the vertices v_1 and v_2.

LEMMA 4. *Let G_1 and G_2 be graphs and let $v_i \in V(G_i)$ for $i = 1, 2$. The graph $G_1 \oplus G_2$ is T-codable if and only if G_1 and G_2 are T-codable. The graph $G_1 \oplus G_2/(v_1, v_2)$ is T-codable if and only if there are T-codings c^1 and c^2 of G_1 and G_2 with the property that*

$$H(c^1(v_1), c^1(x)) + H(c^2(v_2), c^2(y)) > T$$

for

$$x \in V(G_1) - \{v_1\} \qquad \textit{and} \qquad y \in V(G_2) - \{v_2\}.$$

PROOF. Let c be a T-coding of $G_1 \oplus G_2$. Then for $i = 1$ or 2 the function $c^i : V(G_i) \to Q$ given by

$$c^i(x) = c((x, i))$$

is a T-coding of G_i.

Conversely, if c^i is a T-coding of G_i for $i = 1$ or 2 then the function $c : V(G_1 \oplus G_2) \to Q$ given by

$$c_j((x,i)) = \begin{cases} 1 & \text{if} \quad j \leqslant T+1 \quad \text{and} \quad i = 1 \\ 0 & \text{if} \quad j \leqslant T+1 \quad \text{and} \quad i = 2 \\ c_k^1(x) & \text{if} \quad j = T+2k, \quad k \geqslant 1 \quad \text{and} \quad i = 1 \\ 0 & \text{if} \quad j = T+2k, \quad k \geqslant 1 \quad \text{and} \quad i = 2 \\ 0 & \text{if} \quad j = T+1+2k, \quad k \geqslant 1 \quad \text{and} \quad i = 1 \\ c_k^2(x) & \text{if} \quad j = T+1+2k, \quad k \geqslant 1 \quad \text{and} \quad i = 2 \end{cases}$$

is a T-coding of $G_1 \oplus G_2$.

Now suppose that c is a T-coding of $G_1 \oplus G_2/(v_1, v_2)$. Then for $i = 1$ or 2 the function $c^i : V(G_i) \to Q$ defined by

$$c^i(x) = \begin{cases} c((x, i)) & \text{if} \quad x \neq v_i \\ c(0) & \text{if} \quad x = v_i \end{cases}$$

is a T-coding of G_i. Furthermore, for $x \in V(G_1) - \{v_i\}$ and $y \in V(G_2) - \{v_2\}$ we have

$$\begin{aligned} H(c^1(x), c^1(v_1)) + H(c^2(y), c^2(v_2)) &= H(c((x, 1)), c(0)) + H(c((y, 2)), c(0)) \\ &\geqslant H(c((x, 1)), c((y, 2))) > T. \end{aligned}$$

Conversely, suppose that c^1 and c^2 are T-codings of G_1 and G_2 which have the required property. Then the function $c : V(G_1 \oplus G_2/(v_1, v_2)) \to Q$ given by

$$c_{2j-1}(0) = c_j^1(v_1)$$

$$c_{2j}(0) = c_j^2(v_2)$$

$$c_{2j-1}((x, i)) = \begin{cases} c_j^1(x) & \text{if} \quad i = 1 \\ c_j^1(v_1) & \text{if} \quad i = 2 \end{cases}$$

$$c_{2j}((x, i)) = \begin{cases} c_j^2(v_2) & \text{if} \quad i = 1 \\ c_j^2(x) & \text{if} \quad i = 2 \end{cases}$$

is a T-coding of $G_1 \oplus G_2/(v_1, v_2)$.

LEMMA 5. *Let γ, s and N be positive integers with $s \geqslant \max(2, \gamma)$. There is a $2s - 1$ coding c of $G(\gamma, N)$ with the property that $H(c(u), c(x)) \geqslant 2s - \gamma - 1$*

for each vertex x of $G(\gamma, N)$ with $x \neq u$. There is a $4s$ coding $\bar{c}$ of $G(\gamma, N)$ with the property that $H(\bar{c}(u), \bar{c}(x)) \geqslant 4s - 2\gamma$ for each vertex x of $G(\gamma, N)$ with $x \neq u$.

PROOF. For each vertex x of $G(\gamma, N)$ let $I(x)$ be a subset of the positive integers. We may choose these set s so that they have the following properties: $I(x)$ and $I(y)$ are disjoint for $x \neq y$. $| I(u) | = s - \gamma$, $| I(v) | = s - 1$ and $| I(\alpha) | = s$ for $\alpha \in \mathscr{S}(\gamma, N)$.

Since $s \geqslant 2$, $I(x)$ is nonempty for $x \neq u$. Let i_x be the least element of $I(x)$ for $x \neq u$.

We define a function c from the vertices of $G(\gamma, N)$ to Q as follows:

$$c_i(u) = 0 \qquad \text{for all} \quad i$$

$$c_i(v) = \begin{cases} 1 & \text{if} \quad i \in I(u) \cup I(v) \\ 0 & \text{otherwise.} \end{cases}$$

If $\alpha \in \mathscr{S}(\gamma, N)$,

$$c_i(\alpha) = \begin{cases} 1 & \text{if} \quad i \in I(u) \cup I(\alpha) \\ 1 & \text{if} \quad i = i_\beta \quad \text{where} \quad \beta \leqslant \alpha \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that c is a $2s - 1$ coding of $G(\gamma, N)$ with

$$H(c(u), c(x)) \geqslant 2s - \gamma - 1 \qquad \text{for} \qquad x \neq u.$$

Let V be the number of vertices of $G(\gamma, N)$. Let i be a $1 - 1$ mapping of the vertices of $G(\gamma, N)$ onto the integers $1, 2, \ldots, V$. Define a function $\bar{c}$ from the vertices of $G(\gamma, N)$ to Q as follows:

$$\bar{c}_{V+2j-1}(x) = \bar{c}_{V+2j}(x) = c_j(x) \qquad \text{for} \qquad j = 1, 2, \ldots .$$

For $1 \leqslant i \leqslant V$,

$$\bar{c}_i(x) = \begin{cases} 1 & \text{if} \quad i = i(x) \\ 0 & \text{if} \quad i \neq i(x). \end{cases}$$

If x and y are distinct vertices of $G(\gamma, N)$ then

$$H(\bar{c}(x), \bar{c}(y)) = 2H(c(x), c(y)) + 2.$$

The required properties for $\bar{c}$ follow from this relation and the corresponding properties for c.

For each positive integer γ let $G(\gamma)$ denote the graph

$$G(\gamma, 2^{(4\gamma+2)}(4\gamma + 2) + 1)$$

and let

$$\bar{G}(\gamma) = G(\gamma + 1) \oplus G(\gamma)/(u, u).$$

LEMMA 6. *For* $\gamma \geqslant 2$, $T_0(G(\gamma)) = 2\gamma - 1$ *and* $T_e(G(\gamma)) = 4\gamma - 2$. *For* $\gamma \geqslant 1$, $T_0(\bar{G}(\gamma)) = 2\gamma + 3$ *and* $T_e(\bar{G}(\gamma)) = 4\gamma + 4$.

PROOF. Let $\gamma \geqslant 2$. Taking $s = \gamma$ in Lemma 5 we see that $G(\gamma)$ is $2\gamma - 1$ codable. Hence, by Lemma 1, $G(\gamma)$ is $4\gamma - 2$ codable. Therefore $T_0(G(\gamma)) \leqslant 2\gamma - 1$ and $T_e(G(\gamma)) \leqslant 4\gamma - 2$. Suppose $T_e(G(\gamma)) < 4\gamma - 2$. Then, by Lemma 1, $G(\gamma)$ has a $4\gamma - 4$ coding c. Let

$$\alpha, \beta \in \mathscr{S}(\gamma, 2^{(4\gamma+2)}(4\gamma + 2) + 1)$$

with $|\alpha| = |\beta| = 1$ and $\alpha \neq \beta$. By Lemmas 2 and 3,

$$H(c(u), c(\alpha)) \leqslant 4\gamma - 4 - 2(\gamma - 1) = 2\gamma - 2.$$

Similarly, $H(c(u), c(\beta)) \leqslant 2\gamma - 2$. Therefore,

$$H(c(\alpha), c(\beta)) \leqslant H(c(\alpha), c(u)) + H(c(u), c(\beta)) \leqslant 4\gamma - 4.$$

But α and β are not adjacent in $G(\gamma)$ so this contradicts the assumption that c is $4\gamma - 4$ coding. Hence, $T_e(G(\gamma)) = 4\gamma - 2$. By Lemma 1,

$$2T_0(G(\gamma)) \geqslant T_e(G(\gamma)) = 4\gamma - 2$$

so we must also have $T_0(G(\gamma)) = 2\gamma - 1$.

Now let $\gamma \geqslant 1$. Taking $s = \gamma + 2$ in Lemma 5, we see that there are $2\gamma + 3$ codings c' and c'' of $G(\gamma)$ and $G(\gamma + 1)$ with the property that

$$H(c'(u), c'(x)) \geqslant 2\gamma + 4 - \gamma - 1 = \gamma + 3 \qquad \text{for} \qquad x \in V(G(\gamma)) - \{u\}$$

and

$$H(c''(u), c''(x)) \geqslant 2\gamma + 4 - (\gamma + 1) - 1 = \gamma + 2$$

for

$$x \in V(G(\gamma + 1)) - \{u\}.$$

Now $\gamma + 3 + \gamma + 2 > 2\gamma + 3$ so by Lemma 4 the graph

$$\bar{G}(\gamma) = G(\gamma + 1) \oplus G(\gamma)/(u, u).$$

is $2\gamma + 3$ codable.

Now apply the second part of Lemma 5 to $G(\gamma)$ and $G(\gamma + 1)$ with $s = \gamma + 1$. Since

$$4(\gamma + 1) - 2\gamma + 4(\gamma + 1) - 2(\gamma + 1) = 4\gamma + 6 > 4\gamma + 4,$$

it now follows from Lemma 4 that $\bar{G}(\gamma)$ is $4\gamma + 4$ codable.

We have now shown that $T_0(\bar{G}(\gamma)) \leqslant 2\gamma + 3$ and $T_e(\bar{G}(\gamma)) \leqslant 4\gamma + 4$. Suppose $T_e(\bar{G}(\gamma)) < 4\gamma + 4$. Then, by Lemma 1, $\bar{G}(\gamma)$ is $4\gamma + 2$ codable. By Lemma 4 there are $4\gamma + 2$ codings c' and c'' of $G(\gamma)$ and $G(\gamma + 1)$ such that $H(c'(u), c'(v)) + H(c''(u), c''(v)) > 4\gamma + 2$. But by Lemma 3

$$H(c'(u), c'(v)) + H(c''(u), c''(v)) \leqslant 4\gamma + 2 - 2\gamma + 4\gamma + 2 - 2(\gamma + 1)$$
$$= 4\gamma + 2.$$

This contradiction shows that $T_e(\bar{G}(\gamma)) = 4\gamma + 4$. By Lemma 1,

$$2\gamma + 2 = \tfrac{1}{2}\, T_e(\bar{G}(\gamma)) \leqslant T_0(\bar{G}(\gamma)) \leqslant 2\gamma + 3.$$

Since $T_0(\bar{G}(\gamma))$ is odd, we must have $T_0(\bar{G}(\gamma)) = 2\gamma + 3$.

For each positive integer n let $K(n)$ be the graph defined as follows: The vertices of $K(n)$ are the elements of the set $\{0, 1\} \times \{1, 2,..., n\}$. If $\epsilon = 0$ or 1 and $1 \leqslant i \leqslant n$ then the vertex (ϵ, i) is adjacent to every vertex except $(1 - \epsilon, i)$.

LEMMA 7. *The graph $K(1)$ is T-codable for every positive integer T. If n and T are positive integers and $K(n)$ is T-codable, then $K(n)$ and $K(n + 1)$ are $T + 1$ codable.*

PROOF. Let $a, b \in Q$ with $H(a, b) > T$. Let $c((0, 1)) = a$ and $c((1, 1)) = b$. Then c is a T-coding of $K(1)$.

Now let c be a T-coding of $K(n)$. We will assume that c is chosen so that the set

$$I(c) = \{i \mid c_i(x) = 1 \text{ for some vertex } x \text{ of } K(n)\}$$

has as few elements as possible.

Suppose that $H(c(0, j), c(1, j)) \geqslant T + 2$ for $1 \leqslant j \leqslant n$. Let $i_0 \in I(c)$. Define a function $\hat{c} : V(K(n)) \to Q$ by

$$\hat{c}_i(x) = \begin{cases} c_i(x) & \text{if} \quad i \neq i_0 \\ 0 & \text{if} \quad i = i_0 . \end{cases}$$

We have $| I(\hat{c}) | = | I(c) | - 1$ so $\hat{c}$ cannot be a T-coding of $K(n)$. If x and y are vertices of $K(n)$ then $H(\hat{c}(x), \hat{c}(y)) \leqslant H(c(x), c(y)) \leqslant T$ if x and y are adjacent and $H(\hat{c}(x), \hat{c}(y)) \geqslant H(c(x), c(y)) - 1 \geqslant T + 1$ if x and y are not adjacent. Hence, $\hat{c}$ must fail to be a T-coding because it is not a $1 - 1$ function.

Let (δ, i) and (ϵ, j) be distinct vertices of $K(n)$ with $\hat{c}((\delta, i)) = \hat{c}((\epsilon, j))$. Then $H(c((\delta, i)), c((\epsilon, j))) = 1$. Since $(\delta, i) \neq (\epsilon, j)$, (ϵ, j) and $(1 - \delta, i)$ are adjacent. Hence,

$$T + 2 \leqslant H(c(\delta, i), c(1 - \delta, i)) \leqslant H(c(\delta, i), c(\epsilon, j)) + H(c(\epsilon, j), c(1 - \delta, i)),$$
$$\leqslant T + 1.$$

This contradiction shows that for some j with $1 \leqslant j \leqslant n$ we have $H(c(0,j), (c(1,j)) = T + 1$. Without the loss of generality we may assume that $H(c(0, n), c(1, n)) = T + 1$.

We now define a $T + 1$ coding $\bar{c}$ of $K(n + 1)$. The restriction of $\bar{c}$ to the vertices of $K(n)$ will provide a $T + 1$ coding of $K(n)$.

For $1 \leqslant j \leqslant n + 1$, let $\bar{j} = \min(j, n)$. For $\epsilon = 0$ or 1 and $1 \leqslant j \leqslant n + 1$, let

$$\bar{c}_1(\epsilon, j) = \begin{cases} \epsilon & \text{if} \quad j \leqslant n \\ 1 - \epsilon & \text{if} \quad j = n + 1. \end{cases}$$

For $i > 1$ let

$$\bar{c}_i(\epsilon, j) = c_{i-1}(\epsilon, \bar{j}).$$

We first show that $\bar{c}$ is $1 - 1$. Suppose $\bar{c}(\delta, i) = \bar{c}(\epsilon, j)$. Then $c(\delta, \bar{\imath}) = c(\epsilon, \bar{j})$ so $(\delta, \bar{\imath}) = (\epsilon, \bar{j})$. Now $\bar{c}_1(\epsilon, j) = \bar{c}_1(\delta, j)$ so either $i, j \leqslant n$ or $i = j = n + 1$. This together with $\bar{\imath} = \bar{j}$ implies that $i = j$ so $(\delta, i) = (\epsilon, j)$.

We have $H(\bar{c}(\delta, i), \bar{c}(1 - \delta, i)) = 1 + H(c(\delta, \bar{\imath}), c(1 - \delta, \bar{\imath})) > 1 + T$. Now suppose that $H(\bar{c}(\delta, i), \bar{c}(\epsilon, j)) > T + 1$. Then

$$H(c(\delta, \bar{\imath}), c(\epsilon, \bar{j})) \geqslant H(\bar{c}(\delta, i), \bar{c}(\epsilon, j)) - 1 > T$$

so $\bar{\imath} = \bar{j}$ and $\epsilon = 1 - \delta$. If $i \neq j$ then without loss of generality we may assume that $i = n$ and $j = n + 1$. But then we have

$$\begin{aligned} T + 1 < H(\bar{c}(\delta, i), \bar{c}(\epsilon, j)) &= H(c(\delta, n), c(1 - \delta, n)) \\ &\qquad + | \bar{c}_1(\delta, n) - \bar{c}_1(1 - \delta, n + 1) | \\ &= H(c(0, n), c(1, n)) = T + 1. \end{aligned}$$

This contradiction shows that $i = j$ so (δ, i) and $(\epsilon, j) = (1 - \delta, i)$ are nonadjacent.

We have now established that $H(\bar{c}(x), \bar{c}(y)) > T + 1$ if and only if x and y are nonadjacent so $\bar{c}$ is a $T + 1$-coding.

LEMMA 8. *Let T be a positive integer. If $n \geqslant (2T + 1)^{T+1}$ then $K(n)$ is not T-codable.*

PROOF. Suppose the lemma is false. Then there are integers n and T with $n \geqslant (2T + 1)^{T+1}$ and $T > 0$ and a T-coding c of $K(n)$. For each integer k with $0 \leqslant k \leqslant T$ we will construct an element x^k of Q and a subset I_k of the integers $\{1, 2,..., n\}$ with the properties that $| I_k | \geqslant (2T + 1)^{k+1}$ and $H(x^k, c(0, i)) \leqslant k$ for $i \in I_k$.

We begin with $k = T$ and proceed by induction. Let $I_T = \{1, 2,..., n\}$ and $x^T = c(0, 1)$. Suppose x^k and I_k have been defined for some k with $0 < k \leqslant T$. Let $i_0 \in I_k$. Let

$$A = \{j \mid c_j(0, i_0) \neq x_j^k\} \qquad \text{and} \qquad B = \{j \mid c_j(1, i_0) \neq x_j^k\}.$$

We have

$$| A | = H(c(0, i_0), x^k) \leqslant k. \tag{1}$$

If we let $i \in I_k - \{i_0\}$ then

$$| B | = H(c(1, i_0), x^k) \leqslant H(c(1, i_0), c(0, i)) + H(c(0, i), x^k) \leqslant T + k \leqslant 2T. \tag{2}$$

Finally, using (1), we get

$$| B | + k \geqslant | B | + | A | \geqslant | (A \cup B) - (A \cap B) | = H(c(0, i_0), c(1, i_0)) \geqslant T + 1$$

so

$$| B | \geqslant T + 1 - k. \tag{3}$$

For each $j \in B$ let

$$D_j = \{i \in I_k \mid c_j(0, i) \neq x_j^k\}$$

and let y^j be the element of Q defined by

$$y_1^j = \begin{cases} x_\ell^k & \text{if} \quad \ell \neq j \\ 1 - x_\ell^k & \text{if} \quad \ell = j \end{cases}$$

Then for each $i \in D_j$ we have $H(c(0, i), y^j) = H(c(0, i), x^k) - 1 \leqslant k - 1$. Hence, if $| D_j | \geqslant (2T + 1)^k$ for some $j \in B$ we may take $I_{k-1} = D_j$ and $x^{k-1} = y^j$.

Suppose $| D_j | \leqslant (2T + 1)^k - 1$ for each $j \in B$. Let

$$D = \{i \in I_k - \{i_0\} \mid c_j(0, i) = x_j^k \text{ for all } j \in B\}.$$

Then using (2) we have

$$\begin{aligned}(2T + 1)^{k+1} \leqslant | I_k | &\leqslant \sum_{j \in B} | D_j | + | D | + | \{i_0\} | \\ &\leqslant | B | [(2T + 1)^k - 1] + | D | + 1 \\ &\leqslant 2T[(2T + 1)^k - 1] + | D | + 1 \\ &\leqslant 2T(2T + 1)^k + | D |\end{aligned}$$

so $| D | \geqslant (2T + 1)^k$. Let $i \in D$. Using (3) we have

$$\begin{aligned}T &\geqslant H(c(0, i), c(1, i_0)) \\ &= \sum_{j \in B} | c_j(0, i) - c_j(1, i_0) | + \sum_{j \notin B} | (c_j(0, i) - c_j(1, i_0) | \\ &= | B | + H(c(0, i), x^k) \\ &\geqslant T + 1 - k + H(c(0, i), x^k).\end{aligned}$$

Therefore, $H(c(0, i), x^k) \leqslant k - 1$ so we may take $I_{k-1} = D$ and $x^{k-1} = x^k$.

Since $|I_0| \geqslant (2T+1)^{0+1} = 2T+1 \geqslant 3$, there are distinct elements $i, j \in I_0$. Now $H(c(0,i), x^0) = H(c(0,j), x^0) = 0$ so $c(0,i) = x^0 = c(0,j)$. This contradicts the assumption that c is a T-coding and establishes the lemma.

Let γ be a positive integer. By Lemmas 7 and 8 the set of positive integers n for which $K(n)$ is γ-codable is nonempty and bounded. Let n_γ be the largest integer for which $K(n)$ is γ-codable. Let $H(\gamma) = K(n_\gamma)$.

LEMMA 9. *Let T and γ be positive integers. The graph $H(\gamma)$ is T-codable if and only if $T \geqslant \gamma$.*

PROOF. $H(\gamma)$ is γ-codable by definition. By Lemma 7, $H(\gamma)$ is $\gamma+1$ codable. Now, by Lemma 1, $H(\gamma)$ is T-codable for every $T \geqslant \gamma$.

Now suppose that $H(\gamma)$ is T-codable for some T with $0 < T < \gamma$. Then by applying Lemma 7 $\gamma - T$ times we see that $K(n_\gamma + \gamma - T)$ is γ-codable. But $n_\gamma + \gamma - T > n_\gamma$ so this contracts the fact n_γ is the largest integer n for which $K(n)$ is γ-codable.

We are now ready to complete the proof of the theorem. Let T_0 and T_e be positive integers which are respectively odd and even and which satisfy

$$T_0 - 1 \leqslant T_e \leqslant 2T_0. \tag{*}$$

We must exhibit a graph G with $T_0(G) = T_0$ and $T_e(G) = T_e$. We may rewrite (*) as

$$\tfrac{1}{2} T_e \leqslant T_0 \leqslant T_e + 1. \tag{**}$$

If $T_e = 2$ then $T_0 = 1$ or 3 and, by Lemma 9, we may take $G = H(1)$ or $H(2)$. if $T_e = 4$ then $T_0 = 3$ or 5 and we may take $G = H(3)$ or $H(4)$.

Now suppose that $T_e \geqslant 6$. We will first assume that $T_e = 4\gamma - 2$ where $\gamma \geqslant 2$. Then (**) becomes

$$2\gamma - 1 \leqslant T_0 \leqslant 4\gamma - 1$$

so $T_0 = 2\gamma + 2s - 1$ where $0 \leqslant s \leqslant \gamma$.

Let G and H be graphs. Then, by Lemma 4, Lemma 1 and the definition of $T_0(G)$ and $T_e(G)$ we have $T_0(G \oplus H) = \max(T_0(G), T_0(H))$ and $T_e(G \oplus H) = \max(T_e(G), T_e(H))$. Let $G = G(\gamma) \oplus H(2\gamma + 2s - 2)$. Then, by Lemmas 6 and 9

$$\begin{aligned} T_0(G) &= \max(T_0(G(\gamma)), T_0(H(2\gamma + 2s - 2))) \\ &= \max(2\gamma - 1, 2\gamma + 2s - 1) \\ &= 2\gamma + 2s - 1 \end{aligned}$$

since $s \geqslant 0$. Similarly,

$$\begin{aligned} T_e(G) &= \max\,(T_e(G(\gamma)),\ T_e(H(2\gamma + 2s - 2))) \\ &= \max\,(4\gamma - 2,\ 2\gamma + 2s - 2) \\ &= 4\gamma - 2, \end{aligned}$$

since $s \leqslant \gamma$.

Now assume that $T_e = 4\gamma + 4$ where $\gamma \geqslant 1$. Then (**) becomes $2\gamma + 2 \leqslant T_0 \leqslant 4\gamma + 5$ so $T_0 = 2\gamma + 2s + 3$ where $0 \leqslant s \leqslant \gamma + 1$. Let $G = \bar{G}(\gamma) \oplus H(2\gamma + 2s + 2)$. Again by Lemmas 6 and 9 we have

$$\begin{aligned} T_0(G) &= \max\,(T_0(\bar{G}(\gamma)),\ T_0(H(2\gamma + 2s + 2))) \\ &= \max\,(2\gamma + 3,\ 2\gamma + 2s + 3) \\ &= 2\gamma + 2s + 3, \end{aligned}$$

since $s \geqslant 0$. Similarly,

$$\begin{aligned} T_e(G) &= \max\,(T_e(\bar{G}(\gamma)),\ T_e(H(2\gamma + 2s + 2))) \\ &= \max\,(4\gamma + 4,\ 2\gamma + 2s + 2) \\ &= 4\gamma + 4 \end{aligned}$$

since $s \leqslant \gamma + 1$. This completes the proof of the theorem for unconnected graphs.

To extend this result to connected graphs we need one more result. Given graphs G_1 and G_2 and codes c^k for $k = 1, 2$, so that G_k is T_k codable, and let $T = \max\,(T_1, T_2)$. Now the function $c : V(G_1 \oplus G_2) \to Q$ as defined in Lemma 4 gives a T-coding for $G_1 \oplus G_2$.

Let $G_1 \oplus G_2//(i, 1)\,(j, 2)$ be the connected graph formed by joining vertices $(i, 1)$ and $(j, 2)$ of $G_1 \oplus G_2$. Let T_0 and T_e for graphs G_1, G_2 and $G_1 \oplus G_2//(i, I)\,(j, 2)$ be $T_0^{\,1}$, $T_e^{\,1}$, $T_0^{\,2}$, $T_e^{\,2}$, and $T_0^{1,2}$, $T_e^{1,2}$, respectively. Since G_k is a subgraph of $G_1 \oplus G_2//(i, 1)\,(j, 2)$, it follows that

$$T_0^{1,2} \geqslant \max\,(T_0^{\,1}, T_0^{\,2}) \qquad \text{and} \qquad T_e^{1,2} \geqslant \max\,(T_e^{\,1}, T_e^{\,2}).$$

LEMMA 10. *If G_1 and G_2 are T_1 and T_2 codable, respectively, and $T = \max\,(T_1, T_2)$, then $G_1 \oplus G_2//(i, 1)\,(j, 2)$ is T-codable.*

PROOF. Let c^k be a coding for G_k, $k = 1, 2$. Define a new coding $\bar{c}^k$ for G_k as follows.

For all $x \in V(G_1)$

$$\bar{c}_\alpha^{\,1}(x) = \begin{cases} c_\alpha^{\,1}(x) & \text{if} \quad c_\alpha^{\,1}(i) = 0 \\ 1 - c_\alpha^{\,1}(x) & \text{otherwise} \end{cases}$$

and for all $x \in V(G_2)$

$$\bar{c}_\alpha^{\,2}(x) = \begin{cases} c_\alpha^{\,2}(x) & \text{if} \quad c_\alpha^{\,2}(j) = 0 \\ 1 - c_\alpha^{\,2}(x) & \text{otherwise.} \end{cases}$$

Hence $\bar{c}^1(i) = \bar{c}^2(j) = 0$, where 0 is the all zero code, and

$$H(\bar{c}^k(u), \bar{c}^k(v)) = H(c^k(u), c^k(v)) \qquad \text{for} \qquad k = 1, 2.$$

Now from the code defined by the function $c : V(G_1 \oplus G_2//(i, 1)(j, 2)) \to Q$ given by

$$c_\alpha((x, k)) = \begin{cases} 1 & \text{if} \quad \alpha \leqslant T \quad \text{and} \quad k = 1 \\ 0 & \text{if} \quad \alpha \leqslant T \quad \text{and} \quad k = 2 \\ \bar{c}_\beta{}^1(x) & \text{if} \quad \alpha = T + 2\beta, \quad \beta \geqslant 1 \quad \text{and} \quad k = 1 \\ 0 & \text{if} \quad \alpha = T + 2\beta, \quad \beta \geqslant 1 \quad \text{and} \quad k = 2 \\ 0 & \text{if} \quad \alpha = T + 1 + 2\beta, \quad \beta \geqslant 1 \quad \text{and} \quad k = 1 \\ \bar{c}_\beta{}^2(x) & \text{if} \quad \alpha = T + 1 + 2\beta, \quad \beta \geqslant 1 \quad \text{and} \quad k = 2. \end{cases}$$

Now $H(c(i, 1), c(j, 2)) = T$ while $H(c(p, 1), c(q, 2)) \geqslant T + 1$ for all p, q except for the case $p = i$ and $q = j$. Also

$$H(c(u, k), c(v, k)) = H(c^k(u), c^k(v)) \qquad \text{for} \qquad k = 1, 2.$$

Since vertices $(i, 1)$ and $(j, 2)$ are adjacent in $G_1 \oplus G_2//(i, 1)(j, 2)$, the lemma is proven.

As a result of this lemma, we have that

$$T_0^{1,2} = \max(T_0{}^1, T_0{}^2) \qquad \text{and} \qquad T_e^{1,2} = \max(T_e{}^1, T_e{}^2),$$

hence the theorem holds for unconnected and connected graphs.

Reference

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